§3.3 A one-loop computation in Background Field Gauge

Last time we saw

$$g^R = g(1+L_A)^{-1/2}$$

in background field gauge.

Let us compute the renormalization factor L_A at one-loop.

→ take background $A_{\alpha\mu}$ to be space-time independent

→ compute quartic term

Set $A_{\alpha\mu} = \text{const.}$, $\psi = \omega = \omega^* = 0$

$$\mathcal{L}_{MOD} = \mathcal{L} + \mathcal{L}_f + \mathcal{L}_{GH}$$

$$\mathcal{L} = -\frac{1}{4}\left(F_{\alpha\mu\nu} + \bar{D}_\mu A'_{\alpha\nu} - \bar{D}_\nu A'_{\alpha\mu} + C_{\alpha\beta\gamma} A'_{\beta\mu} A'_{\gamma\nu}\right)^2$$
$$- \bar{\psi}'(\bar{\not{D}} - i t_\alpha \not{A}'_\alpha + m)\psi',$$

$$\mathcal{L}_f = -\frac{1}{2\xi} f_\alpha f_\alpha = -\frac{1}{2\xi}\left(\bar{D}_\mu A'^{\mu}_\alpha\right)^2,$$

$$\mathcal{L}_{GH} = -(\bar{D}_\mu \omega'^*_\alpha)(\bar{D}^\mu \omega'_\alpha - C_{\alpha\beta\gamma}\omega'_\beta A'^{\mu}_\gamma).$$

→ one-loop result is calculated from quadratic terms

$$\mathcal{L}_{QUAD} = -\frac{1}{4}\left(\bar{D}_\mu A'_{\alpha\nu} - \bar{D}_\nu A'_{\alpha\mu}\right)^2 - \frac{1}{2} F_\alpha^{\mu\nu} C_{\alpha\beta\gamma} A'_{\beta\mu} A'_{\gamma\nu}$$

$$- \bar{\psi}'(\bar{\not{D}} + m)\psi' - \frac{1}{2\xi}\left(\bar{D}_\mu A'^\mu_\alpha\right)^2 - \left(\bar{D}_\mu \omega'^*_\alpha\right)\left(\bar{D}^\mu \omega'_\alpha\right)$$

This gives the action

$$I_{QUAD} = \int d^4x\, \mathcal{L}_{QUAD}$$

$$= -\frac{1}{2}\int d^4x\, d^4y\, A'^\mu_\alpha(x)\, A'^\nu_\beta(y)\, \mathcal{D}^A_{x\alpha\mu, y\beta\nu}$$

$$-\int d^4x\, d^4y\, \bar{\psi}'_k(x)\, \psi'_\ell(y)\, \mathcal{D}^\psi_{xk, y\ell} - \int d^4x\, d^4y\, \omega^{*\prime}_\alpha(x)\, \omega'_\beta(y)\, \mathcal{D}^\omega_{x\alpha, y\beta},$$

$\longrightarrow$ one-loop contribution:

$$\exp\left(i\Gamma^{1\text{ loop}}[A]\right) \sim \int_{1PI} \left(\prod dA'\right)\left(\prod d\psi'\right)\left(\prod d\bar{\psi}'\right)\left(\prod d\omega'\right)\left(\prod d\omega^{*\prime}\right)$$

$$(1) \qquad \times \exp\left(i I_{QUAD}\left[A', \psi', \bar{\psi}', \omega', \omega'^*; A\right]\right)$$

$$\sim \left(\mathrm{Det}\,\mathcal{D}^A\right)^{-1/2}\left(\mathrm{Det}\,\mathcal{D}^\psi\right)^{+1}\left(\mathrm{Det}\,\mathcal{D}^\omega\right)^{+1}.$$

$\longrightarrow$ $\mathcal{D}$s can be diagonalized by passing to momentum space:

$$\mathcal{D}_{q\ldots, p\ldots} = \int \frac{d^4x}{(2\pi)^2}\, e^{-iq\cdot x} \int \frac{d^4y}{(2\pi)^2}\, e^{ip\cdot y}\, \mathcal{D}_{x\ldots, y\ldots}$$

With A constant, this gives

$$\mathcal{D}_{q\ldots, p\ldots} = \delta^4(p-q)\, \mathcal{M}_{\ldots, \ldots}(q)$$

where $\mathcal{M}$ are finite q-dependent matrices

For example, one finds

$$\mathcal{M}^{A}_{\alpha\mu,\beta\nu}(q) = \eta_{\mu\nu}\left(-iq_{\lambda}\delta_{\gamma\alpha} + A_{\delta\lambda}C_{\gamma\delta\alpha}\right)\left(iq^{\lambda}\delta_{\gamma\beta} + A^{\lambda}_{\varepsilon}C_{\gamma\varepsilon\beta}\right)$$
$$- \left(-iq_{\nu}\delta_{\gamma\alpha} + A_{\delta\nu}C_{\gamma\delta\alpha}\right)\left(iq_{\mu}\delta_{\gamma\beta} + A_{\varepsilon\mu}C_{\gamma\varepsilon\beta}\right)$$
$$+ F_{\gamma\mu\nu}C_{\gamma\alpha\beta}$$
$$+ \left(-iq_{\mu}\delta_{\gamma\alpha} + A_{\delta\mu}C_{\gamma\delta\alpha}\right)\left(iq_{\nu}\delta_{\gamma\beta} + A_{\varepsilon\nu}C_{\gamma\varepsilon\beta}\right)/\}$$
$$+ \xi\text{-terms},$$

where $F_{\alpha\mu\nu} = C_{\alpha\beta\gamma}A_{\beta\mu}A_{\gamma\nu}$

From eq. (1) we then get

$$i\Gamma^{(1\text{ loop})}[A] = -\frac{1}{2}\ln \mathrm{Det}\,\mathcal{D}^{A} + \ln \mathrm{Det}\,\mathcal{D}^{\psi} + \ln \mathrm{Det}\,\mathcal{D}^{\omega}$$
$$= -\frac{1}{2}\mathrm{Tr}\ln \mathcal{D}^{A} + \mathrm{Tr}\ln \mathcal{D}^{\psi} + \mathrm{Tr}\ln \mathcal{D}^{\omega}$$
$$= \delta^{4}(p-q)\int d^{4}q\left[-\frac{1}{2}\mathrm{tr}\ln \mathcal{M}^{A}(q) + \mathrm{tr}\ln \mathcal{M}^{\psi}(q)\right.$$
$$\left. + \mathrm{tr}\ln \mathcal{M}^{\omega}(q)\right]. \qquad (2)$$

'tr' here instead of 'Tr' denotes the usual traces of finite matrices

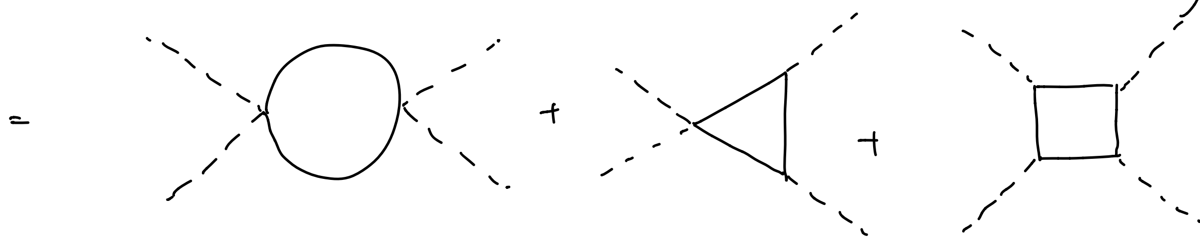
→ isolate terms in (2) of 4th order in A

define $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1 + \mathcal{M}_2$,

where $\mathcal{M}_n$ contains $n = 0, 1,$ or 2 factors of A

then

$$[\mathrm{tr}\ln \mathcal{M}]_{A^4} = \mathrm{tr}\left\{-\tfrac{1}{2}[\mathcal{M}_0^{-1}\mathcal{M}_2]^2 + [\mathcal{M}_0^{-1}\mathcal{M}_1]^2\mathcal{M}_0^{-1}\mathcal{M}_2 - \tfrac{1}{4}[\mathcal{M}_0^{-1}\mathcal{M}_1]^4\right\}$$



Performing the integrations (exercise), we get

- $\int d^4q\,[\mathrm{tr}\ln \mathcal{M}^A(q)]_{A^4} = -\frac{5}{3}\,X\, C_{\gamma\alpha\beta}C_{\delta\alpha\beta}F_{\gamma\mu\nu}F_\delta^{\mu\nu}$

where $\quad X = \int d^4q\,[q^2 - i\varepsilon]^{-2}$

- $\int d^4q\,[\mathrm{tr}\ln \mathcal{M}^\omega(q)]_{A^4} = \frac{1}{12}\,X\, C_{\gamma\alpha\beta}C_{\delta\alpha\beta}F_{\gamma\mu\nu}F_\delta^{\mu\nu}$

- $\int d^4q\,[\mathrm{tr}\ln \mathcal{M}^\psi(q)]_{A^4} = -\frac{1}{3}\,X\, F_{\gamma\mu\nu}F_\delta^{\mu\nu}\,\mathrm{tr}\{t_\gamma t_\delta\}$

Summing up all 3 contributions, one gets

$$\Gamma_{A^4}^{(1\text{ loop})} = \frac{-iX}{(2\pi)^4}\int d^4x\, F_{\gamma\mu\nu}F_\delta^{\mu\nu}\left[\left(\frac{5}{6}+\frac{1}{12}\right)C_{\gamma\alpha\beta}C_{\delta\alpha\beta} - \frac{1}{3}\mathrm{tr}\{t_\gamma t_\delta\}\right],$$

For $SU(N)$ gauge theory with n_f fermions in defining representation, we get

$$C_{\gamma\alpha\beta}C_{\delta\alpha\beta} = g^2 C_1\delta_{\gamma\delta}, \quad \mathrm{tr}\{t_\gamma t_\delta\} = g^2 C_2\delta_{\gamma\delta}$$

$$\text{with } C_1 = N, \quad C_2 = n_f/2$$

$$\rightarrow \Gamma_{A^4}^{(1\ \text{loop})} = \frac{-ig^2 X}{(2\pi)^4}\int d^4x\, F_{\gamma\mu\nu} F_{\gamma}^{\mu\nu}\left[\frac{11}{12}C_1 - \frac{1}{3}C_2\right]$$

By doing a Wick rotation, X can be written as

$$X = i\int_0^{\infty} \frac{2\pi^2 q^3 dq}{q^4} \sim 2\pi^2 i \int_{\mu}^{\Lambda} \frac{dq}{q} = 2\pi^2 i \ln\left(\frac{\Lambda}{\mu}\right)$$

$$\rightarrow L_A = \frac{-g^2}{2\pi^2}\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right)\ln\left(\frac{\Lambda}{\mu}\right) + O(g^4)$$

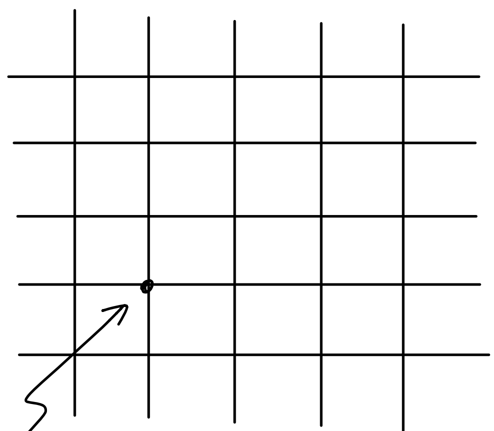
Using, $g_R = (1+L_A)^{-1/2} g$ then gives

$$g_R = g\left[1 + \frac{g^2}{4\pi^2}\ln\left(\frac{\Lambda}{\mu}\right)\left(\frac{11}{12}C_1 - \frac{1}{3}C_2\right) + O(g^4)\right]$$

$\rightarrow$ For $C_2 < \frac{11 C_1}{4}$, the physical coupling g_R "increases" relative to the bare coupling g !

§4. Critical Phenomena and the Renormalization Group

Consider the Ising model on a lattice in d dimensions:



there are N sites
→ 2^N spin configurations

site i carries spin $s_i = \pm 1$

To a given spin configuration, we ascribe an energy $E[s_i] = -\sum_{i,j} J_{ij} s_i s_j - \sum_i h_i s_i$

→ positive values of J give lower energies to configurations with parallel spins (ferromagnetic coupling)
negative values prefer opposite spin configurations (anti-ferremagnetic coupling)

Probability of a state is proportional to:

$$P[s_i] = \exp\left(-\beta E[s_i]\right) = \exp\left(\sum_{i,j} K_{ij}\, s_i s_j + \sum H_i s_i\right)$$

where $K_{ij} = \beta J_{ij}$, $H_i = \beta h_i$, $\beta = (kT)^{-1}$,

k is the Boltzmann constant, T temperature

Partition function is given by

$$Z[H_i] = \sum_{\{s_i\}} \exp\left(-\beta E[s_i]\right)$$

$Z[H_i]$ generates all correlation functions:

$$M_i = \langle s_i \rangle_{H_i=0} = Z^{-1} \left.\frac{\partial Z}{\partial H_i}\right|_{H=0}$$

is average magnetization at site i.

Symmetry of energy under $s_i \mapsto -s_i$ implies $M=0$, when $H=0$.

If, $J_{ij} = J_{i-j}$, $H_i = H$, the system is translationally invariant.

$\to \langle s_i \rangle = \langle s \rangle$ independent of i

and we have

$$M(H) = \frac{1}{N} Z^{-1} \frac{\partial Z}{\partial H} = \frac{1}{N} \sum_i \langle s_i \rangle$$

The susceptibility is given as

$$\chi(H) = \frac{\partial M}{\partial H}$$

and $\chi = \left.\frac{\partial M}{\partial H}\right|_{H=0} = \frac{1}{N} \sum_{i,j} \left(\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right)$

$$= \frac{1}{N} \sum_{i,j} \langle (s_i - M)(s_j - M) \rangle$$

Define $g(r_i - r_j) = \langle (s_i - \langle s_i \rangle)(s_j - \langle s_j \rangle) \rangle$

$$= Z^{-1} \frac{\partial^2 Z}{\partial H_i \partial H_j} - \left(Z^{-1} \frac{\partial Z}{\partial H_i} \right)^2$$

Using translational invariance, we can rewrite

$$\chi = \sum_{\vec{R}} g(\vec{R}),$$

where $\vec{R}$ are all vectors of the lattice relative to one given site.

The mean energy is

$$\langle H \rangle = - \frac{\partial}{\partial \beta} \ln Z[0]$$

and the specific heat is

$$C = \frac{1}{N} \frac{\partial \langle E \rangle}{\partial T} = \frac{1}{N} \beta^2 \frac{\partial^2}{\partial \beta^2} \ln Z[0]$$

$$= \beta^2 \sum_{\vec{R}} g_E(\vec{R}), \quad g_E(\vec{R}) = \langle (E_0 - \langle E_0 \rangle)(E_R - \langle E_R \rangle) \rangle$$